A note on heterogeneous decompositions into spanning trees

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Abstract

In answer to a question of Eggleton, we prove that the complete multigraph on 5 vertices with edge multiplicity 6, namely $K_5^{(6)}$, has a decomposition into 5 copies of the family of trees of order 5 and that $K_7^{(22)}$ has a decomposition into 7 copies of the family of trees of order 7. We prove something similar for K_{2n+1} for $n \leq 13$.

1 Introduction and definitions

Let $\mathfrak{T}(n)$ denote the family of trees of order n. Let $\tau(n) = |\mathfrak{T}(n)|$. Note that this is Sloane's A000055; see [S]. Eggleton has recently shown [E] that the complete multigraph $K_6^{(2)}$ of order 6 with edge multiplicity 2 has a decomposition into the elements of $\mathfrak{T}(6)$. He asks whether $K_5^{(6)}$ has a decomposition into 5 copies of $\mathfrak{T}(5)$ and whether $K_7^{(22)}$ has a decomposition into 7 copies of $\mathfrak{T}(7)$. We answer these questions in the affirmative and prove some similar results not mentioned by Eggleton.

Definition 1. *Let n, s, and t be natural numbers. Then*

$$dc_n(s,t) = \begin{cases} |s-t| \ \textit{if} \ |s-t| \leq \frac{n}{2} \\ \\ n - |s-t| \ \textit{if} \ |s-t| \geq \frac{n}{2} \end{cases}$$

Definition 2. *Let T be a tree of order $2n+1$. If it is possible to label the vertices of T uniquely with the numbers $1, \ldots, 2n+1$ so that the induced*

edge labels $dc_{2n+1}(\ell(u_e), \ell(v_e))$ *comprise the multiset* $\{1, 1, 2, 2, \ldots, n, n\}$ *then we say that* T *is* semigraceful*.*

This definition is of course inspired by Rosa's celebrated:

Definition 3. *A tree of order* p *is* graceful *when the vertices can be labeled* $0, \ldots, p-1$ *in such a way that the induced edge labels* $|\ell(u_e) - \ell(v_e)|$ *are distinct.*

See [R] and [G] for details. Rosa's purpose in making the definition was related to decompositions of the complete graph into trees, although it has since acquired a vivid and independent life. Note that semigraceful labelings have a certain family resemblance to the various flavors of equitable labelings, for an introduction to which see [G].

2 Results

The following is really more of an observation than a lemma:

Lemma 1. *If a tree of odd order is graceful then it is semigraceful.*

However there are semigraceful labelings of odd trees which are not also graceful labelings. For instance label the vertices of P_5, the path of order 5, with $2, 3, 1, 4, 5$.

Theorem 1. *If* T *is a semigraceful tree of order* $2n+1$ *then* $K_{2n+1}^{(2)}$ *is (cyclically) decomposable into* $2n+1$ *copies of* T*.*

Proof. Draw $K_{2n+1}^{(2)}$ with its vertices evenly spaced around a circle. Label the vertices $1, \ldots, 2n+1$ in cyclic order. Embed T into $K_{2n+1}^{(2)}$ as directed by the semigraceful vertex labels. This embedding of T uses 2 edges of each possible cyclic distance in $K_{2n+1}^{(2)}$, so that when T is rotated cyclically by one step n times, each edge of $K_{2n+1}^{(2)}$ is used exactly once. $\square$

Corollary 1. *If every element of* $\mathfrak{T}(2n+1)$ *is semigraceful then* $K_{2n+1}^{(2\tau(2n+1))}$ *has a decomposition into* $2n+1$ *copies of* $\mathfrak{T}(2n+1)$*.*

Aldred and McKay [A] have shown that every tree of order ≤ 27 is graceful, and hence semigraceful. From this follows the affirmative answer to Eggleton's question about $K_5^{(6)}$ and $K_7^{(22)}$, as well as the fact that $K_{2n+1}^{(2\tau(2n+1))}$ has such a decomposition for all $n \leq 13$. Now, it seems that for the most part $gcd(2n+1, \tau(2n+1)) = 1$ (speaking nontechnically,

that is) , and hence that $2\tau(2n+1)$ is the least possible edge multiplicity which will allow for such a decomposition. However, $gcd(21, \tau(21)) = gcd(21, 2144505) = 3$ and $gcd(25, \tau(25)) = gcd(25, 104636890) = 5$. In these cases it is possible, at least as far as edge-counting goes, that $K_{21}^{(1429670)}$ is decomposable into 7 copies of $\mathfrak{T}(21)$ and that $K_{25}^{(41854756)}$ is decomposable into 5 copies of $\mathfrak{T}(25)$. We conjecture that such decompositions exist.

References

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